ALL SAINTS' COLLEGE

Mathematics Specialist
Test 3 2016

Vectors in 3D

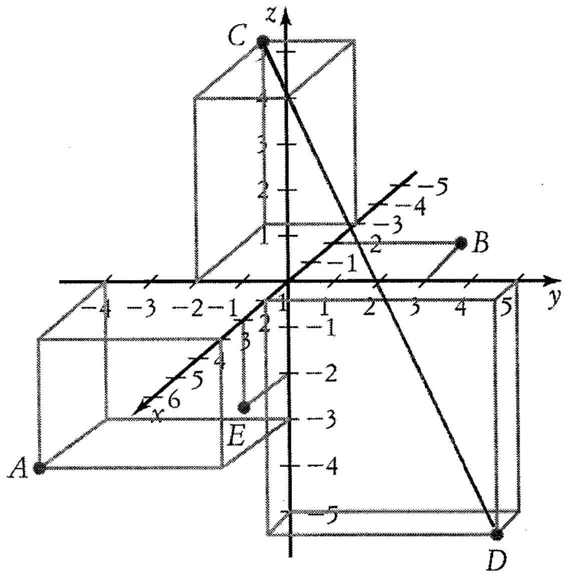
NAME: Solutions (ML4) TEACHER: MLA

45 marks 45 minutes

One unfolded A4 page of notes, SCSA formulae booklet and ClassPad calculator permitted

Question 1 [2 marks]

Use the diagram below to determine the (exact) distance from C to D:



$C = (-3, -2, 4)$

$D = (1, 5, -5)$

$\vec{CD} = \underset{\sim}{d} - \underset{\sim}{c}$

$= \begin{bmatrix} 4 \\ 7 \\ -9 \end{bmatrix}$

$|\vec{CD}| = \sqrt{146}$ units.

Question 2 [1, 1 & 1 = 3 marks]

Vector **u** has norm 8, and both azimuthal (in the x-y plane) and altitude angles of 60°.

Vector $\mathbf{v} = 2\mathbf{i} - \mathbf{j} + 4\mathbf{k}$

(a) State the (exact) rectangular form for **u**

ClassPad: $[8, \angle 60, \angle 30] \Rightarrow \underset{\sim}{u} = \begin{bmatrix} 2 \\ 2\sqrt{3} \\ 4\sqrt{3} \end{bmatrix}$

(b) Find the acute angle between **u** and **v**, correct to two decimal places.

ClassPad: Angle $\left(\begin{bmatrix} 2 \\ 2\sqrt{3} \\ 4\sqrt{3} \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} \right) = 39.60°$

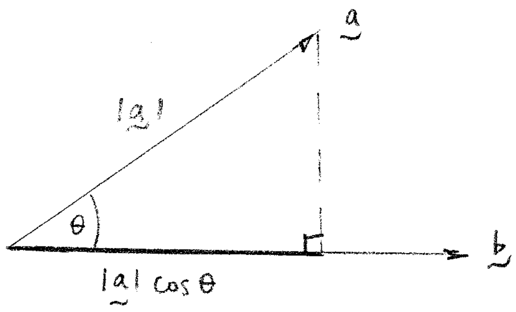
(c) Determine the (exact) vector orthogonal to both **u** and **v**

ClassPad: crossP $\left(\begin{bmatrix} 2 \\ 2\sqrt{3} \\ 4\sqrt{3} \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} \right) = \begin{bmatrix} 12\sqrt{3} \\ 8\sqrt{3} - 8 \\ -4\sqrt{3} - 2 \end{bmatrix}$

Question 3 [4 marks]

Find the projection of $\mathbf{a} = (-4, 2, -3)$ on $\mathbf{b} = (2, -5, 1)$

projection of $\underset{\sim}{a}$ on $\underset{\sim}{b}$ $= |\underset{\sim}{a}| \cos\theta$



projection (scalar)

$$|\underset{\sim}{a}| \cos\theta = |\underset{\sim}{a}| \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}$$

$$= \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{b}|}$$

$$= \frac{\begin{bmatrix} -4 \\ 2 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}}{\sqrt{4 + 25 + 1}}$$

$$= \frac{-8 - 10 - 3}{\sqrt{30}}$$

$$= \frac{-21}{\sqrt{30}}$$

$$= \frac{-7\sqrt{30}}{10}$$

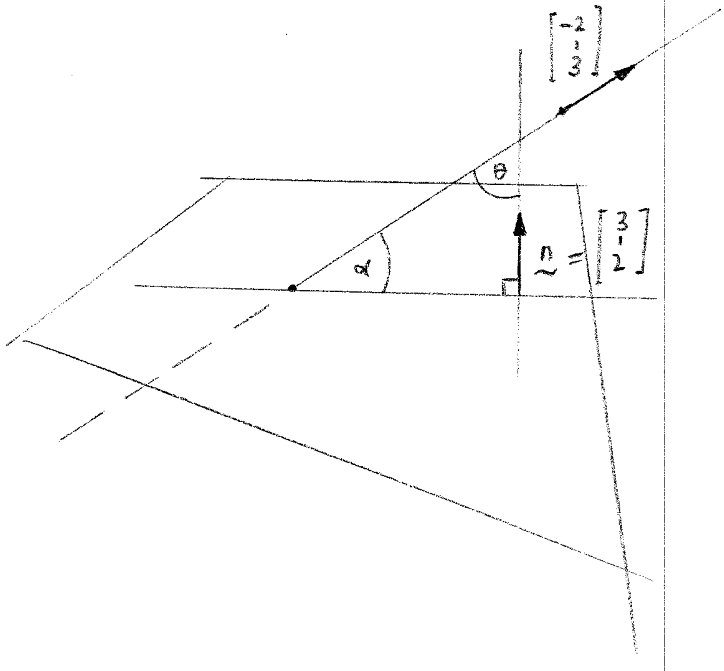
Question 4 [4 marks]

C is the midpoint of the line segment AB. D is a point **not** on the line AB such that DC = CA.

Use vector methods to prove that DA is perpendicular to DB.

α is the required angle

$\underset{\sim}{r} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + \lambda \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$



θ = angle between vectors $\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$

$= \text{Angle}\left(\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}\right)$

$= 85.9°$ (1dp)

$\therefore \alpha = 90 - 85.9$

$= 4.1°$ (1dp)

Alternatively: using $\underset{\sim}{a} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$ and $\underset{\sim}{b} = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$ and dot product:

$$\cos\theta = \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}$$

$$= \frac{-6 + 1 + 6}{\sqrt{14} \times \sqrt{14}}$$

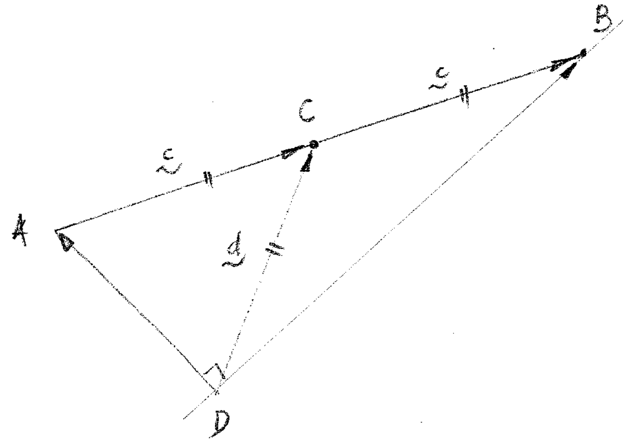
$$= \frac{1}{14}$$

$\Rightarrow \theta = 85.9°$ (1dp)

$\therefore \alpha = 4.1°$ (1dp)

Question 5 [4 marks]

Find the angle between the line $\mathbf{r} = \begin{bmatrix} 1 - 2\lambda \\ \lambda \\ 2 + 3\lambda \end{bmatrix}$ and the plane $\mathbf{r} \cdot \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} = -4$



RTP $\vec{DA} \cdot \vec{DB} = 0$

LHS $= \vec{DA} \cdot \vec{DB}$

$= (\underset{\sim}{d} - \underset{\sim}{c}) \cdot (\underset{\sim}{d} + \underset{\sim}{c})$

$= \underset{\sim}{d} \cdot \underset{\sim}{d} + \underset{\sim}{d} \cdot \underset{\sim}{c} - \underset{\sim}{d} \cdot \underset{\sim}{c} - \underset{\sim}{c} \cdot \underset{\sim}{c}$

$= |\underset{\sim}{d}|^2 - |\underset{\sim}{c}|^2$

$= 0 \quad \because |\underset{\sim}{d}| = |\underset{\sim}{c}|$

Question 6 [5 marks]

Find the equation of the plane containing the points E (4, −3, 1), F (5, 0, 2) and G (3, 2, 5)

$\vec{EF} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$ $\vec{GF} = \begin{bmatrix} 2 \\ -2 \\ -3 \end{bmatrix}$ These are vectors in the plane

$\vec{EF} \times \vec{GF} = \begin{bmatrix} -7 \\ 5 \\ -8 \end{bmatrix} = \underset{\sim}{n}$ = Vector normal to the plane

So, $\underset{\sim}{r} \cdot \begin{bmatrix} -7 \\ 5 \\ -8 \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -7 \\ 5 \\ -8 \end{bmatrix}$

$\underset{\sim}{r} \cdot \begin{bmatrix} -7 \\ 5 \\ -8 \end{bmatrix} = -51$ or $-7x + 5y - 8z = -51$

Alternatively: $\underset{\sim}{r} = \begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + \mu \begin{bmatrix} 2 \\ -2 \\ -3 \end{bmatrix}$

Alternatively: $n_1 (x - a_1) + n_2 (y - a_2) + n_3 (z - a_3) = 0$

where $\underset{\sim}{n} = \begin{bmatrix} -7 \\ 5 \\ -8 \end{bmatrix}$ and $\underset{\sim}{a} = \begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix}$

So, $-7(x-4) + 5(y+3) - 8(z-1) = 0$

$-7x + 28 + 5y + 15 - 8z + 8 = 0$

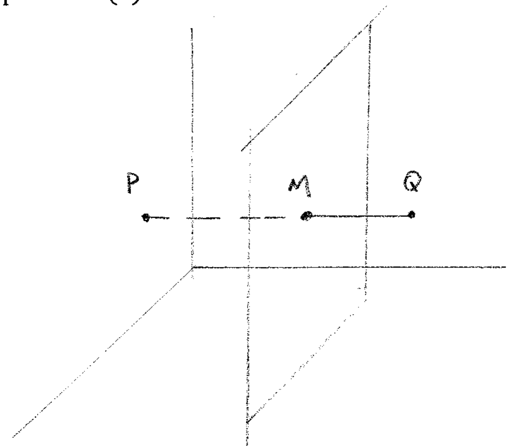
$-7x + 5y - 8z = -28 - 15 - 8$

$\therefore -7x + 5y - 8z = -51$

Question 7 [4 & 1 = 5 marks]

(a) Determine the vector equation of the plane consisting of all points that are equidistant from the points P $(-1, 2, -3)$ and Q $(4, -2, 2)$.

(b) Hence, or otherwise, state the Cartesian form of the plane in (a)



(a) M = midpoint

$= \frac{1}{2}(-1+4,\ 2-2,\ -3+2)$

$= \frac{1}{2}(3, 0, -1)$

$= \left(\frac{3}{2}, 0, \frac{-1}{2}\right)$

= point on the plane

$\vec{PQ}$ = perpendicular to required plane

$= \underset{\sim}{q} - \underset{\sim}{p}$

$= \begin{bmatrix} 4+1 \\ -2-2 \\ 2+3 \end{bmatrix}$

$= \begin{bmatrix} 5 \\ -4 \\ 5 \end{bmatrix}$

$= \underset{\sim}{n}$

So, $\underset{\sim}{r} \cdot \begin{bmatrix} 5 \\ -4 \\ 5 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ 0 \\ -\frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} 5 \\ -4 \\ 5 \end{bmatrix}$

$\underset{\sim}{r} \cdot \begin{bmatrix} 5 \\ -4 \\ 5 \end{bmatrix} = 5$

(b) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} 5 \\ -4 \\ 5 \end{bmatrix} = 5 \Rightarrow 5x - 4y + 5z = 5$

(b) Alternatively: Cartesian equation of plane is given as

$n_1(x-a_1) + n_2(y-a_2) + n_3(z-a_3) = 0$

where $\underset{\sim}{n} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \\ 5 \end{bmatrix}$ and $\underset{\sim}{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ 0 \\ -\frac{1}{2} \end{bmatrix}$

So, $5\left(x - \frac{3}{2}\right) - 4(y-0) + 5\left(z + \frac{1}{2}\right) = 0$

$5x - \frac{15}{2} - 4y + 5z + \frac{5}{2} = 0$

$\therefore \quad 5x - 4y + 5z = 5$

Question 8 [5 & 2 = 7 marks]

Use elementary row operations to reduce the following system of equations to echelon form:

$$x + y + (k - 3)z = 1$$

$$2x + 4y + 4z = 3$$

$$-x + y + (7 - 2k)z = m$$

$$\left[\begin{array}{ccc|c} 1 & 1 & k-3 & 1 \\ 2 & 4 & 4 & 3 \\ -1 & 1 & 7-2k & m \end{array}\right] \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & k-3 & 1 \\ 0 & 2 & 10-2k & 1 \\ 0 & 2 & 4-k & m+1 \end{array}\right] \begin{array}{l} R_1 \\ R_2 - 2R_1 = R_4 \\ R_3 + R_1 = R_5 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & k-3 & 1 \\ 0 & 2 & 10-2k & 1 \\ 0 & 0 & k-6 & m \end{array}\right] \begin{array}{l} R_1 \\ R_4 \\ R_5 - R_4 = R_6 \end{array}$$

(a) State the values of k and m if the system is to have:

(i) A unique solution $k \neq 6 \; ; \; m \in \mathbb{R}$

(ii) Infinite solutions $k = 6 \; ; \; m = 0$

(iii) No solutions $k = 6 \; ; \; m \neq 0$

(b) In the case of infinite solutions, find the particular solution for which x = 1.

If $k = 6$ and $m = 0$ $\Rightarrow$ $\left[\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 0 & 2 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $\rightarrow$ infinite solutions.

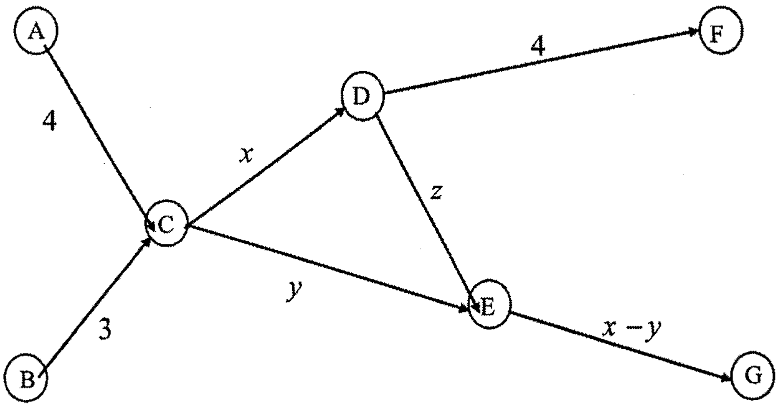
when $x = 1$, solve $\left.\begin{array}{l} 1 + y + 3z = 1 \\ 2y - 2z = 1 \end{array}\right\}$ simultaneous equations.

ClassPad : $\left(1, \frac{3}{8}, -\frac{1}{8}\right)$.

Question 9 [6 marks]

The schematic diagram below shows the volume of passengers (in tens of thousands) through airports A, B, C, D, E, F and G in the month of January.

Use Gaussian elimination to find the values of x, y and z if there are 10 000 less arrivals than departures at C, an equal number of arrivals and departures at D, and 20 000 more departures than arrivals at E.



$x + y = 8$ (1)

$4 + z = x \Rightarrow x - z = 4$ (2)

$x - y = y + z + 2 \Rightarrow x - 2y - z = 2$ (3)

Form an augmented matrix :

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 8 \\ 1 & 0 & -1 & 4 \\ 1 & -2 & -1 & 2 \end{array}\right] \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 8 \\ 0 & -1 & -1 & -4 \\ 0 & -3 & -1 & -6 \end{array}\right] \begin{array}{l} R_1 \\ R_2 - R_1 = R_4 \\ R_3 - R_1 = R_5 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 8 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 2 & 6 \end{array}\right] \begin{array}{l} R_1 \\ R_4 \\ R_5 - 3R_4 = R_6 \end{array}$$

So, $2z = 6 \therefore z = 3$

$y + 3 = 4 \therefore y = 1$

$x + 1 = 8 \therefore x = 7$

Solution : $(7, 1, 3)$

Question 10 [5 marks]

At time t = 0, the position (**r**) and velocity (**v**) vectors of a British warship (B) and German submarine (G) are as follows:

$$\mathbf{r}_B = \begin{bmatrix} 1150 \\ 827 \\ 0 \end{bmatrix} \text{m} \; ; \; \mathbf{v}_B = \begin{bmatrix} 10 \\ -2 \\ 0 \end{bmatrix} \text{m/sec} \qquad \mathbf{r}_G = \begin{bmatrix} 1345 \\ 970 \\ 0 \end{bmatrix} \text{m} \; ; \; \mathbf{v}_G = \begin{bmatrix} -5 \\ -13 \\ -4 \end{bmatrix} \text{m/sec}$$

warship travelling on surface

submarine diving

If both vessels maintain their respective velocities, determine the time at which the submarine is directly beneath the warship, and the depth of the submarine at this instant.

Note. The x-y plane represents the surface of the ocean

- Both vessels initially ($t=0$) on the surface. ie. z-components are zero.
- x and y components will be identical when warship is directly above submarine.

$$\underset{\sim}{r}_B(t) = \begin{bmatrix} 1150 \\ 827 \\ 0 \end{bmatrix} + t \begin{bmatrix} 10 \\ -2 \\ 0 \end{bmatrix} \quad \text{and} \quad \underset{\sim}{r}_G(t) = \begin{bmatrix} 1345 \\ 970 \\ 0 \end{bmatrix} + t \begin{bmatrix} -5 \\ -13 \\ -4 \end{bmatrix}$$

So, solve $1150 + 10t = 1345 - 5t \Rightarrow t = 13$

$827 - 2t = 970 - 13t \Rightarrow t = 13$

That is, submarine is directly beneath warship when $t = 13$ seconds

- Location of submarine at time $t = \begin{bmatrix} 1345 - 5(13) \\ 970 - 13(13) \\ 0 - 4(13) \end{bmatrix} = \begin{bmatrix} 1280 \\ 801 \\ -52 \end{bmatrix}$ metres

That is, submarine's depth is 52 metres.

End of Test 3